

LOCAL OSCILLATIONS IN MODERATELY DENSE SEQUENCES OF PRIMES

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To Robert Tichy, on the occasion of his 60th birthday

ABSTRACT. The distribution of differences of consecutive members of sequences of primes is investigated. A quantitative measure for oscillations among these differences is the curvature of the sequence. If the sequence is not too sparse, then sharp estimates for its curvature are provided.

1. INTRODUCTION

In an influential paper, Erdős and Turán [2] showed that when (p_n) denotes the sequence of all prime numbers arranged in increasing order, then there are infinitely many sign changes among the numbers

$$(1) \quad p_{n+1}^2 - p_n p_{n+2}.$$

Motivated by quantitative versions of this result due to Rényi [8] and Erdős and Rényi [1], we develop this theme further in the context of sequences that are not too sparse.

Theorem 1. *Let \mathcal{P} be a set of primes with the property that*

$$(2) \quad \frac{(\log x)^{4/3}}{x} \#\{p \in \mathcal{P} : p \leq x\}$$

tends to infinity with x . If p_n denotes the enumeration of the set \mathcal{P} in increasing order, then the sequence (1) changes sign infinitely often.

Our main object of study is the *curvature* of sequences. The idea is due to Rényi [8]. Consider at least three distinct points z_1, \dots, z_N in the complex plane. With the argument of a complex number chosen in the interval $(-\pi, \pi]$, the sum

$$(3) \quad \sum_{n=1}^{N-2} \left| \arg \frac{z_{n+2} - z_{n+1}}{z_{n+1} - z_n} \right|.$$

is referred to as the total curvature of the polygonal line connecting z_{n-1} with z_n for $2 \leq n \leq N$, because this adds up the (non-negative) angles between the line segments from z_n to z_{n+1} , and on to z_{n+2} . For a set of primes \mathcal{P} , again enumerated in increasing order as p_n , we take $z_n = n + i \log p_n$, and then let $K_N(\mathcal{P})$ denote the sum in (3) with this special choice of z_n . This is the curvature of \mathcal{P} , truncated at N .

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Now suppose that we knew that $K_N(\mathcal{P})$ were unbounded. Then, if the segment $(\log p_n)_{n_0 \leq n \leq N}$ is either concave or convex, then $K_N(\mathcal{P}) - K_{n_0}(\mathcal{P}) \leq \frac{1}{2}\pi$ which is impossible for large N . We conclude that the sequence

$$\log p_{n+2} - 2 \log p_{n+1} + \log p_n$$

changes sign infinitely often, and on taking exponentials this is the same as exhibiting sign changes in the sequence (1). In particular, Theorem 1 will follow once we have established that $K_N(\mathcal{P})$ is unbounded for the sequences of primes satisfying (2). Further, we see that the growth rate of $K_N(\mathcal{P})$ is a rough measure for the oscillations in the sequence (1).

Rényi [8] in 1950 considered the sequence of all primes and bounded their curvature, hereafter denoted by K_N , from below by

$$K_N \gg \log \log \log N.$$

Shortly afterwards, in collaboration with Erdős [1] (see also [7]) he determined the order of magnitude of K_N , now showing that

$$(4) \quad \log N \ll K_N \ll \log N.$$

Their methods rely on the prime number theorem. Our concern in this paper is with estimates for the curvature that are based solely on lower bounds for the number of primes in a given sequence, such as in (2). Before we can formulate our principal estimate, we have to set up some notation.

We work relative to an arithmetic progression. When $a, q \in \mathbb{N}$ with $1 \leq a \leq q$ and $(a, q) = 1$, let $\mathcal{P}_{q,a}$ denote the set of all primes $p \equiv a \pmod{q}$. We refer to a subset $\mathcal{P} \subset \mathcal{P}_{q,a}$ as *dense* if there are positive numbers δ and x_0 with the property that whenever $x \geq x_0$, then

$$(5) \quad \#\{p \in \mathcal{P} : p \leq x\} \geq \delta \pi(x; q, a)$$

where as usual $\pi(x; q, a)$ is the number of primes not exceeding x in $\mathcal{P}_{q,a}$. More generally, if $\delta : [3, \infty) \rightarrow (0, 1]$ is monotonically decreasing with $\delta(x) \geq (\log x)^{-1}$, and (5) is satisfied with $\delta = \delta(x)$ for all $x \geq x_0$, then¹ the set \mathcal{P} is called δ -dense (relative to x_0 and $\mathcal{P}_{q,a}$). The lower bound on δ ensures that \mathcal{P} is an infinite set, enumerated in ascending order by p_n , as before. Then $K_N(\mathcal{P})$ is defined for all $N \geq 3$. We also put $\delta_N = \delta(p_N)$.

Theorem 2. *Fix a number $x_0 \geq 3$ and a decreasing function $\delta : [3, \infty) \rightarrow (0, 1]$ with $\delta(x) \geq (\log x)^{-1}$ for all $x \geq 3$. Then there is a sequence of natural numbers $N_0(q)$ with the property that for all $N \geq N_0(q)$ and for all sets of primes \mathcal{P} that are δ -dense relative to x_0 and some $\mathcal{P}_{q,a}$, one has*

$$(6) \quad K_N(\mathcal{P}) \leq 500 \delta_N^{-1} \log N.$$

If $\delta(x)^2 \log x$ tends to infinity with x , then one also has

$$K_N(\mathcal{P}) \geq 10^{-8} \delta_N^3 \log N.$$

Theorem 2 may be applied to the arithmetic progression $\mathcal{P}_{q,a}$ itself, with $\delta = 1$. We then conclude as follows.

¹It may seem unnatural to include the lower bound on δ in this definition, but more rapidly decaying functions will play no role in this paper, and it simplifies the exposition later that δ is not too small, *a fortiori*.

Corollary. *With $N_0(q)$ as in the preceding Theorem, for $N \geq N_0(q)$ one has*

$$10^{-8} \log N \leq K_N(\mathcal{P}_{q,a}) \leq 500 \log N.$$

This contains (4) as a very special case. Note that here as well as in Theorem 2 no effort has been made to optimise the numerical constants.

When δ decays it is important to have at hand a lower bound for δ_N . One has

$$(7) \quad \delta_N \geq \delta(4\varphi(q)N(\log N)^2)$$

for all large N . We show this in passing, in Section 3 below. In particular, if δ is a decreasing function such that $\delta(x)^3 \log x$ tends to infinity with x and \mathcal{P} is δ -dense, then by (7) and Theorem 2 we see that $K_N(\mathcal{P})$ does not remain bounded. Hence, Theorem 1 is merely a corollary of Theorem 2.

We are not aware of earlier results of the type considered in Theorem 1 or Theorem 2 for sequences that are not quite dense. For other developments of the ideas deriving from [1, 2, 8] see Pomerance [6].

With the sequence of primes comprising \mathcal{P} we associate their *second differences*

$$(8) \quad \Delta_n = p_{n+2} - 2p_{n+1} + p_n.$$

Following Rényi in spirit, our approach to Theorem 2 rests on the observation that Δ_n is not too small for many values of n . Our next theorem is a strong quantitative version of this principle.

Theorem 3. *Fix x_0 and δ as in Theorem 2. Then there is a sequence of natural numbers $N_0(q)$ with the property that for all $N \geq N_0(q)$ and for all sets of primes \mathcal{P} that are δ -dense relative to x_0 and some $\mathcal{P}_{q,a}$, one has*

$$\sum_{N < n \leq 2N} \frac{|\Delta_n|}{p_n} \leq \frac{11}{\delta_{2N+2}}.$$

If $\delta(x)^2 \log x$ tends to infinity with x , then one also has

$$\sum_{N < n \leq 2N} \frac{|\Delta_n|}{p_n} \geq 10^{-7} \delta_{2N}^3.$$

Perhaps it is worth remarking that the upper bound recorded in Theorem 3 is nearly the best possible. We demonstrate this with a scattered sequence that we briefly discuss at the end of the paper.

The proof of Theorem 3 invokes upper bounds for the number of triplets of primes that come close to an arithmetic 3-progression. In Section 2 we use Selberg's sieve and a method of Gallagher [3] to manufacture a suitable estimate, but it is worth pointing out that the older Brun's sieve and a technique of Hardy and Littlewood [5] would yield results of comparable strength. Equipped with the sieve estimates, the transition to the lower bound announced in Theorem 3 is elementary, and is performed in Section 3. For Theorem 2, we need a more explicit version of Theorem 3 (see Lemma 4 below) and the method of Rényi [8]. The latter depends, in its original form, on the prime number theorem and is therefore not directly applicable to subsets of the primes. In Section 4, we reconfigure Rényi's approach and establish Theorem 2. Thus our arguments that have elements in common with the work of Erdős and Rényi [1], rely on methods that have been familiar for decades and yet, are of strength sufficient to address sequences of primes that are not quite dense.

If more is known about the distribution of the sequence (p_n) , then our arguments sometimes produce estimates that are superior to those recorded in the theorems. For example this is the case when the δ -dense set \mathcal{P} has the additional property that the numbers p_{2n}/p_n remain bounded. With this extra assumption, the factor δ^{-1} can be deleted from the upper bounds in Theorems 2 and 3. For more details on this refinement, the reader is referred to sections 3 and 4 below.

2. A SIEVE ESTIMATE

In this section we establish an auxiliary estimate concerned with triplets of primes. The main result is Lemma 2 below, and this depends on a certain singular series average that we now describe.

Throughout this section, let $\mathbf{h} = (h, h') \in \mathbb{N}^2$ and suppose that $h' < h$. Then

$$(9) \quad D = hh'(h - h')$$

is an even natural number. For a prime p , let $\nu_{\mathbf{h}}(p)$ denote the number of distinct residue classes, modulo p , in which the numbers $0, h, h'$ lie. Then $1 \leq \nu_{\mathbf{h}}(2) \leq 2$ and $1 \leq \nu_{\mathbf{h}}(p) \leq 3$ for all odd primes p . Further, it is immediate that one has $\nu_{\mathbf{h}}(p) = 3$ if and only if $p \nmid D$. For a given $q \in \mathbb{N}$, we now define the *singular product*

$$(10) \quad \mathfrak{S}_{q, \mathbf{h}} = \prod_{p \nmid q} \left(1 - \frac{\nu_{\mathbf{h}}(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-3}.$$

Note that

$$(11) \quad \left(1 - \frac{\nu_{\mathbf{h}}(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-3} = 1 + a(p, \nu_{\mathbf{h}}(p))$$

where

$$a(p, \nu) = \frac{p^3 - (p-1)^3 - \nu p^2}{(p-1)^3}.$$

One readily checks that for odd primes one has

$$(12) \quad |a(p, \nu)| \leq 3/(p-1) \quad (\nu = 1, 2), \quad |a(p, 3)| \leq 4/(p-1)^2.$$

Hence, recalling that $\nu_{\mathbf{h}}(p) = 3$ holds for all $p \nmid D$, one finds that the product (10) converges absolutely to a non-negative limit.

Lemma 1. *Let $\varepsilon > 0$. Then, uniformly for $q \in \mathbb{N}$ and $0 < \alpha \leq 1$ one has*

$$\sum_{1 \leq h \leq H} \sum_{|2h' - h| < \alpha h} \mathfrak{S}_{q, \mathbf{h}} = \frac{1}{2} \alpha H^2 + O(H^{1+\varepsilon}).$$

When $q = 1$ a similar estimate occurs in Gallagher [3], but the average there is over more parameters, and is more symmetric. We therefore give a complete proof, although we shall follow [3] quite closely. For convenience, it is appropriate to put

$$\mathcal{H} = \{(h, h') \in \mathbb{N}^2 : h \leq H, |2h' - h| < \alpha h\}.$$

Note that $\alpha \leq 1$ ensures that for any pair $(h, h') \in \mathcal{H}$ one has $h' < h$. In particular, $\mathfrak{S}_{q, \mathbf{h}}$ is defined. Now put

$$(13) \quad a_{\mathbf{h}}(r) = \prod_{p|r} a(p, \nu_{\mathbf{h}}(p)).$$

Then, by (11), the absolutely convergent product in (10) can be rewritten as

$$(14) \quad \mathfrak{S}_{q, \mathbf{h}} = \sum_{\substack{r=1 \\ (r, q)=1}}^{\infty} \mu(r)^2 a_{\mathbf{h}}(r).$$

From (12), (13) and a familiar divisor estimate, we infer that $a_{\mathbf{h}}(r) \ll r^{\varepsilon-2}(r, D)$. Consequently, since $\mathbf{h} \in \mathcal{H}$ implies $0 < D \leq H^3$, we conclude that

$$\sum_{r > H} \mu(r)^2 |a_{\mathbf{h}}(r)| \ll H^{\varepsilon-1}$$

holds uniformly in $\mathbf{h} \in \mathcal{H}$. Then, the crude bound $\#\mathcal{H} \leq H^2$ and (14) suffice to deduce that

$$(15) \quad \sum_{\mathbf{h} \in \mathcal{H}} \mathfrak{S}_{q, \mathbf{h}} = \sum_{\substack{r \leq H \\ (r, q)=1}} \mu(r)^2 \sum_{\mathbf{h} \in \mathcal{H}} a_{\mathbf{h}}(r) + O(H^{1+\varepsilon}).$$

Note that this estimate is uniform with respect to α and q .

Consider the inner sum over \mathbf{h} in (15) for a given square-free number $r \leq H$. Let $r = p_1 \cdots p_{\omega}$ be the prime factorization. We apply (13) and sort the $\mathbf{h} \in \mathcal{H}$ according to given values of $\nu_{\mathbf{h}}(p_j)$ ($1 \leq j \leq \omega$) to conclude that

$$(16) \quad \sum_{\mathbf{h} \in \mathcal{H}} a_{\mathbf{h}}(r) = \sum_{\substack{1 \leq \nu_j \leq 3 \\ 1 \leq j \leq \omega}} a(p_1, \nu_1) \cdots a(p_{\omega}, \nu_{\omega}) S(r, \boldsymbol{\nu})$$

where $S(r, \boldsymbol{\nu})$ is the number of $\mathbf{h} \in \mathcal{H}$ with $\nu_{\mathbf{h}}(p_j) = \nu_j$ for all $1 \leq j \leq \omega$. Note that the condition $\nu_{\mathbf{h}}(p_j) = \nu_j$ depends only on the residue classes of h and h' , modulo p_j . Hence, we may arrange h and h' into residue classes, modulo r , and then apply the Chinese Remainder Theorem to see that

$$S(r, \boldsymbol{\nu}) = \sum_{\substack{1 \leq a, a' \leq r \\ \nu_{\mathbf{a}}(p_j) = \nu_j \\ 1 \leq j \leq \omega}} \sum_{\substack{\mathbf{h} \in \mathcal{H} \\ h \equiv a \pmod{r} \\ h' \equiv a' \pmod{r}}} 1.$$

For $r \leq H$, we also have

$$\sum_{\substack{\mathbf{h} \in \mathcal{H} \\ h \equiv a \pmod{r} \\ h' \equiv a' \pmod{r}}} 1 = \sum_{\substack{1 \leq h \leq H \\ h \equiv a \pmod{r}}} \left(\frac{\alpha h}{r} + O(1) \right) = \frac{1}{2} \alpha \left(\frac{H}{r} \right)^2 + O\left(\frac{H}{r} \right).$$

Now let $t(p, \nu)$ denote the number of choices for a, a' with $1 \leq a, a' \leq p$ such that the numbers $0, a, a'$ lie in exactly ν residue classes, modulo p . Then, again by the Chinese Remainder Theorem,

$$\sum_{\substack{1 \leq a, a' \leq r \\ \nu_{\mathbf{a}}(p_j) = \nu_j \\ 1 \leq j \leq \omega}} 1 = \prod_{j=1}^{\omega} t(p_j, \nu_j),$$

and on collecting together we infer that

$$S(r, \boldsymbol{\nu}) = \left(\frac{1}{2} \alpha \left(\frac{H}{r} \right)^2 + O\left(\frac{H}{r} \right) \right) \prod_{j=1}^{\omega} t(p_j, \nu_j).$$

Now (16) delivers

$$(17) \quad \sum_{\mathbf{h} \in \mathcal{H}} a_{\mathbf{h}}(r) = \frac{1}{2} \alpha \left(\frac{H}{r} \right)^2 \prod_{p|r} \sum_{\nu=1}^3 a(p, \nu) t(p, \nu) + O\left(\frac{H}{r} \prod_{p|r} \sum_{\nu=1}^3 |a(p, \nu)| t(p, \nu) \right).$$

An inspection of the definition of $t(p, \nu)$ readily shows that

$$t(p, 1) = 1, \quad t(p, 2) = 3(p-1), \quad t(p, 3) = (p-1)(p-2).$$

A short calculation leads to the identity

$$\sum_{\nu=1}^3 a(p, \nu) t(p, \nu) = 0$$

for all primes p , and for odd primes, by (12) we also have

$$\sum_{\nu=1}^3 |a(p, \nu)| t(p, \nu) \leq 15.$$

It follows that the leading term in (17) vanishes except when $r = 1$. Moreover, again using a divisor estimate, we see that the error term in (17) does not exceed $O(Hr^{\varepsilon-1})$. Hence, by (15),

$$\sum_{\mathbf{h} \in \mathcal{H}} \mathfrak{S}_{q, \mathbf{h}} = \frac{1}{2} \alpha H^2 + O(H^{1+\varepsilon}) + O\left(H \sum_{r \leq H} r^{\varepsilon-1} \right),$$

and the conclusion of Lemma 1 follows.

Lemma 2. *Suppose that $0 < \alpha \leq 1 \leq H \leq x$ and that $a, q \in \mathbb{N}$ are coprime with $1 \leq a \leq q$. Let $U = U_{\alpha, q, a}(x, H)$ denote the number of primes p, p', p'' with $p \equiv p' \equiv p'' \equiv a \pmod{q}$ that satisfy the inequalities*

$$(18) \quad 5 \leq p \leq x, \quad p < p'' \leq p + qH, \quad |p'' - 2p' + p| < \alpha(p'' - p).$$

Further let $\varepsilon > 0$. Then there are a number $x_2 = x_2(q)$ depending only on q and a number $E = E_\varepsilon$ depending only on ε such that whenever $x \geq x_2$ one has

$$(19) \quad U \leq (25\alpha H^2 + EH^{1+\varepsilon}) \frac{x}{\varphi(q)(\log x)^3}.$$

Proof. Suppose that p, p', p'' is a triple counted by U . We write

$$(20) \quad p = a + ql, \quad p'' - p = qh, \quad p' - p = qh'.$$

Then $l \in \mathbb{N}_0$, $(h, h') \in \mathbb{N}^2$, and the conditions (18) imply that

$$(21) \quad 0 \leq l \leq x/q, \quad h \leq H, \quad |h - 2h'| \leq \alpha h.$$

By (20), it follows that U does not exceed the number of $l \in \mathbb{N}_0$, $(h, h') \in \mathbb{N}^2$ satisfying (21) and $a + ql \geq 5$ for which the three numbers

$$(22) \quad a + ql, \quad a + q(l + h), \quad a + q(l + h')$$

are all prime.

Let $V(h, h') = V(\mathbf{h})$ denote the number of integers l with $0 \leq l \leq x/q$ and $a + ql \geq 5$ for which the numbers (22) are simultaneously prime. Then, in the notation of the proof of Lemma 1, the above argument shows that

$$U \leq \sum_{\mathbf{h} \in \mathcal{H}} V(\mathbf{h}).$$

Further, the quantity $V(\mathbf{h})$ is readily estimated by an upper bound sieve. We wish to apply [4, Theorem 5.7], and with this end in view we consider, for a prime p , the number $\varrho_{\mathbf{h}}(p)$ of incongruent solutions in z of the congruence

$$(a + qz)(a + q(z + h))(a + q(z + h')) \equiv 0 \pmod{p}.$$

Then, whenever $p \mid q$, one has $\varrho_{\mathbf{h}}(p) = 0$ while in the contrary case $p \nmid q$ it is immediate that $\varrho_{\mathbf{h}}(p) = \nu_{\mathbf{h}}(p)$. If $\mathbf{h} \in \mathcal{H}$ is such that $\varrho_{\mathbf{h}}(p) < p$ holds for all primes p , then [4, Theorem 5.7] is applicable and delivers the inequality

$$(23) \quad V(\mathbf{h}) \leq 50 \mathfrak{S}_{q, \mathbf{h}} \frac{x}{\varphi(q)} (\log x)^{-3}$$

for all x that are sufficiently large in terms of q , as one readily confirms by inspecting (10) and the Euler product in [4, (5.8.3)].

It remains to evaluate $V(\mathbf{h})$ in those cases where $\varrho_{\mathbf{h}}(p) < p$ fails for some prime p . The trivial upper bound $\varrho_{\mathbf{h}}(p) \leq \min(3, p)$ shows that this is possible only when $p = 2$ or 3 . Further, the hypothesis that $\varrho_{\mathbf{h}}(2) = 2$ implies that $2 \nmid q$, and that at least one of h, h' is odd. By (21) we then find that one of the differences $p' - p$, $p'' - p$ is odd which is impossible for $p \geq 5$. This shows that $\varrho_{\mathbf{h}}(2) = 2$ implies $V(\mathbf{h}) = 0$, and a similar argument confirms that the same is true when $\varrho_{\mathbf{h}}(3) = 3$. In particular, we now see that (23) holds for all $\mathbf{h} \in \mathcal{H}$. Summing (23) over these \mathbf{h} with the aid of Lemma 1 yields Lemma 2.

3. SECOND DIFFERENCES - PROOF OF THEOREM 3

We launch an attack toward the estimates claimed in Theorem 3 with a preliminary remark. Throughout, suppose that x_0 and δ are fixed, as in Theorem 2. Let \mathcal{P} be a set of primes, choose a, q with $\mathcal{P} \subset \mathcal{P}_{q, a}$, and assume that (5) holds for all $x \geq x_0$. Suppose it were the case that $p_n > n^2$ holds for some n with $n^2 \geq x_0$. Then, in (5) we take $x = n^2$ and use the lower bound for $\delta(x)$ to infer that

$$n \geq \delta(n^2) \pi(n^2; q, a) \geq (2 \log n)^{-1} \pi(n^2; q, a).$$

The prime number theorem in arithmetic progressions supplies a number $x_1(q)$ such that whenever $x \geq x_1(q)$ then one has $\pi(x; q, a) \geq x/(2\varphi(q) \log x)$. Hence, for $n^2 \geq \max(x_0, x_1(q))$, we conclude that

$$n \geq \frac{n^2}{4\varphi(q)(\log n)^2}.$$

This is absurd for n sufficiently large in terms of q . It follows that there is a number n_0 , depending only on x_0 and q , with the property that whenever $n \geq n_0$ then the inequalities

$$(24) \quad p_n \leq n^2 \quad \text{and} \quad \delta_n^{-1} \leq 2 \log n$$

hold. These bounds are improved in the following lemma, but they play a role in its proof.

Lemma 3. *Let x_0, δ, \mathcal{P} and a, q be as in the preceding paragraph. Then there is a number n_0 depending only on x_0 and q such that whenever $n \geq n_0$, one has*

$$\frac{3}{4} \varphi(q) n \log n \leq p_n \leq 2 \varphi(q) \delta_n^{-1} n \log n.$$

Within the proof, we may suppose that (5) holds with $\delta = \delta(x)$. But then, for $x \leq p_n$, the bound (5) also holds with $\delta = \delta_n$. Now suppose for contradiction that $p_n > x_0$ and $p_n > 2\varphi(q)\delta_n^{-1}n \log n$ hold simultaneously. We may use (5) with $x = 2\delta_n^{-1}\varphi(q)n \log n$ and then see that

$$n \geq \pi(2\delta_n^{-1}\varphi(q)n \log n; q, a).$$

Using the prime number theorem in arithmetic progressions much as above, this implies via (24) that

$$n \geq \frac{3}{2}n \frac{\log n}{\log \varphi(q)n}.$$

This is certainly false for n large in terms of q . The upper bound for p_n follows.

Next, let ϖ denote the n -th member of the ascending sequence of *all* primes in $\mathcal{P}_{q,a}$. Then $p_n \geq \varpi$, and by the prime number theorem in arithmetic progressions once again, one has $\varpi \geq \frac{3}{4}\varphi(q)n \log n$ for all large n . This completes the proof of Lemma 3.

The lower bound (7) is now immediate. Indeed, by Lemma 3 and (24), we have

$$\delta_n = \delta(p_n) \geq \delta(2\varphi(q)\delta_n^{-1}n \log n) \geq \delta(4\varphi(q)n(\log n)^2),$$

as required.

The next task ahead of us is to establish Theorem 3. For the upper bound, we apply the triangle inequality to (8) and then see from Lemma 3 that

$$(25) \quad \sum_{N < n \leq 2N} \frac{|\Delta_n|}{p_n} \leq 2 \sum_{N < n \leq 2N+1} \frac{p_{n+1} - p_n}{p_{N+1}} \leq \frac{2p_{2N+2}}{p_{N+1}} \leq \frac{11}{\delta_{2N+2}},$$

provided only that N is large. This already completes the proof of the upper bound, but there is a simple variant of this argument. Suppose the δ -dense set has the additional property that there exist a number A such that

$$(26) \quad \frac{p_{2m}}{p_m} \leq A$$

holds for all large m . Then the inequalities in (25) provide the alternative estimate

$$(27) \quad \sum_{N < n \leq 2N} \frac{|\Delta_n|}{p_n} \leq 2A.$$

This substantiates a remark that we have made in the introductory part of the paper.

The lower bound in Theorem 3 will be deduced from the sieve bounds established in the previous section. It will be useful to introduce the parameters

$$(28) \quad C = 2/\delta_{2N}, \quad B = 10^{-5}C^{-2}.$$

We will use this notation in the remainder of this paper.

Lemma 4. *Fix x_0 and δ as in Theorem 2, and suppose that $\delta(x)^2 \log x$ tends to infinity with x . Then there is a sequence of natural numbers $N_2(q)$ with the property that for all sets of primes \mathcal{P} that are δ -dense relative to x_0 and some $\mathcal{P}_{q,a}$, and for all $N \geq N_2(q)$, the set*

$$\mathcal{B}(N) = \{N < n \leq 2N - 2 : p_{n+2} - p_n \leq 33C\varphi(q) \log N, |\Delta_n| \geq B\varphi(q) \log N\}$$

has at least $N/2$ elements.

Note that once this lemma is established, we may apply Lemma 3 to conclude that

$$\sum_{n \in \mathcal{B}(N)} \frac{|\Delta_n|}{p_n} \geq \frac{1}{4} C^{-1} B > 10^{-7} \delta_{2N}^3$$

holds whenever $N \geq N_2(q)$. This includes the lower bound recorded in Theorem 3.

We now give the proof of Lemma 4. Let \mathcal{S}_1 be the set of all $n \in (N, 2N - 2]$ where $p_{n+2} - p_n > 33C\varphi(q) \log N$. Then

$$\#\mathcal{S}_1 \leq \sum_{N < n \leq 2N} \frac{p_{n+2} - p_n}{33C\varphi(q) \log N} \leq \frac{p_{2N} + p_{2N-1}}{33C\varphi(q) \log N},$$

and by Lemma 3, for sufficiently large N , we conclude that $\#\mathcal{S}_1 \leq N/8$.

Next, let \mathcal{S}_2 be the set of all $n \in (N, 2N - 2]$ where $p_{n+2} - p_n \leq B\varphi(q) \log N$. Then, in view of (8) and Lemma 3, the primes $p = p_n$, $p' = p_{n+1}$ and $p'' = p_{n+2}$ satisfy the conditions (18) with $\alpha = 1$, $x = 2C\varphi(q)N \log 2N$ and $H = \frac{\varphi(q)}{q} B \log N$. Note that (7) implies that $B \log N$ tends to infinity with N . Hence H is large when N is large (in terms of q). Therefore, for large N , Lemma 2 is applicable, and delivers the estimates

$$\begin{aligned} \#\mathcal{S}_2 &\leq U_{1,q,a}(2C\varphi(q)N \log 2N, H) \\ &\leq (25H^2 + E_\varepsilon H^{1+\varepsilon}) \frac{2C\varphi(q)N \log 2N}{(\varphi(q) \log 2N + O(\log \log N))^3} \\ &\leq 100B^2CN \leq 10^{-8}N, \end{aligned}$$

provided again that N is sufficiently large in terms of q .

For notational convenience, put

$$J_0 = B\varphi(q) \log N, \quad J_1 = 33C\varphi(q) \log N,$$

and let \mathcal{S}_3 be the set of all $n \in (N, 2N - 2]$ where

$$J_0 < p_{n+2} - p_n \leq J_1 \quad \text{and} \quad |\Delta_n| < J_0.$$

Note that a number $n \in (N, 2N - 2]$ that is in none of the sets \mathcal{S}_1 , \mathcal{S}_2 , \mathcal{S}_3 lies in $\mathcal{B}(N)$. Hence, once we have proved that $\#\mathcal{S}_3 \leq N/10$ holds for large N , the proof of Lemma 4 will be complete.

We proceed by a dissection argument. Let $\mathcal{S}_3(H)$ be the subset of \mathcal{S}_3 where $\frac{1}{2}qH < p_{n+2} - p_n \leq qH$. The range of H relevant to us is the interval

$$2J_0 \leq qH \leq 2J_1.$$

For such a value of H and $n \in \mathcal{S}_3(H)$ we have $|\Delta_n| \leq J_0 = \alpha(\frac{1}{2}qH) \leq \alpha(p_{n+2} - p_n)$ where we chose

$$\alpha = \frac{2J_0}{qH}.$$

By Lemma 3, we may take $x = 2C\varphi(q)N \log 2N$ in Lemma 2, and then find that

$$\begin{aligned} \#\mathcal{S}_3(H) &\leq U_{\alpha,q,a}(2C\varphi(q)N \log 2N, H) \\ &\leq (25\alpha H^2 + E_\varepsilon H^{1+\varepsilon}) \frac{2CN \log 2N}{(\log(2C\varphi(q)N \log 2N))^3}. \end{aligned}$$

When N is sufficiently large in terms of q , this upper bound simplifies to

$$\#\mathcal{S}_3(H) \leq (50J_0Hq^{-1} + E_\varepsilon H^{1+\varepsilon}) \frac{3CN}{(\log 2N)^2}.$$

We take $H = 2^j J_0 q^{-1}$ with $2^j \leq J_1/J_0 = 33C/B$ and sum over $j \geq 1$. Since \mathcal{S}_3 is covered by the union of these $\mathcal{S}_3(H)$, we conclude that for large N we have

$$\#\mathcal{S}_3 \leq (50J_0J_1q^{-2} + E_\varepsilon(J_1/q)^{1+\varepsilon}) \frac{6CN}{(\log 2N)^2} \leq 10^4 BC^2 N.$$

By hypothesis, we have $10^4 BC^2 = 1/10$ so that indeed $\#\mathcal{S}_3 \leq N/10$, as required.

4. CURVATURE - PROOF OF THEOREM 2

This section is devoted to the proof of Theorem 2. Recall that $z_n = n + i \log p_n$, and with this choice put

$$k_n = \left| \arg \frac{z_{n+1} - z_n}{z_n - z_{n-1}} \right|.$$

By the elementary properties of the arctan function and its addition theorem, one finds the alternative representations

$$\begin{aligned} k_n &= \left| \arctan \log \frac{p_{n+2}}{p_{n+1}} - \arctan \log \frac{p_{n+1}}{p_n} \right| \\ (29) \quad &= \arctan \frac{\left| \log \frac{p_{n+2} p_n}{p_{n+1}^2} \right|}{1 + \left(\log \frac{p_{n+2}}{p_{n+1}} \right) \left(\log \frac{p_{n+1}}{p_n} \right)}. \end{aligned}$$

It will also be useful to put

$$(30) \quad \Gamma_n = (p_{n+2} - p_{n+1})(p_{n+1} - p_n) p_{n+1}^{-2}.$$

Then, by (8), one has

$$(31) \quad \frac{p_{n+2} p_n}{p_{n+1}^2} = 1 + \frac{\Delta_n}{p_{n+1}} - \Gamma_n.$$

This identity also occurs in Rényi's work (see [8], eqn. (5)), and is crucial to his arguments. Before we proceed further, we note that

$$0 < \Gamma_n \leq \frac{p_{n+2} - p_{n+1}}{p_{n+1}}.$$

In particular, Lemma 3 yields $\Gamma_n \leq (p_{n+2} - p_{n+1}) / (\frac{3}{4} \varphi(q) n \log n)$, and the argument that was used in (25) produces

$$(32) \quad \sum_{N < n \leq 2N} \Gamma_n \leq \frac{6}{\delta_{2N+2}},$$

at least when N is large in terms of q , as we assume from now on. For such N we proceed to derive the inequality

$$(33) \quad \sum_{N < n \leq 2N} k_n \leq \frac{300}{\delta_{2N+2}}.$$

Let \mathcal{C} denote the set of all $n \in (N, 2N]$ where $\Gamma_n > 1/8$, and let \mathcal{D} denote the set of all $n \in (N, 2N]$ where $|\Delta_n| > p_n/8$. Finally, let \mathcal{E} denote the set of the remaining n in $(N, 2N]$. By (32), it follows that $\#\mathcal{C} \leq 48/\delta_{2N+2}$, and similarly, one finds from (25) that $\#\mathcal{D} \leq 88/\delta_{2N+2}$. The trivial bound $k_n \leq \pi/2$ suffices to see that the contribution from all $n \in \mathcal{C} \cup \mathcal{D}$ to the sum in (33) does not exceed $68\pi/\delta_{2N+2}$.

Now suppose that $n \in \mathcal{E}$. Then

$$\left| \frac{\Delta_n}{p_{n+1}} - \Gamma_n \right| \leq \frac{1}{4}.$$

Hence, on applying the familiar bound $\arctan t \leq t$ that is valid for all $t \geq 0$, we first see from (29) that

$$k_n \leq \left| \log \frac{p_{n+2}p_n}{p_{n+1}^2} \right|,$$

and then, by observing that for real numbers t with $|t| \leq 1/4$ one has $|\log(1+t)| \leq 2|t|$, we conclude via (31) that

$$k_n \leq 2 \frac{|\Delta_n|}{p_n} + 2\Gamma_n.$$

We sum this over $n \in \mathcal{E}$. Then, by (25) and (32), we see that (33) indeed holds.

The upper bound reported in Theorem 2 is readily deduced from (33). There is a number $N_1(q)$ such that (33) holds for all $N \geq N_1(q)$. But then, if a large M is given, we may take $N = 2^j N_1(q) \leq M$ in (33) and sum over j . Using the trivial bound for $k_n \leq \pi/2$ when $n \leq N_1(q)$ and observing that δ is decreasing, we find in this way that

$$K_M(\mathcal{P}) \leq \frac{\pi}{2} N_1(q) + 499 \frac{\log M}{\delta_M}.$$

When N is sufficiently large in terms of q , this implies the upper bound recorded in Theorem 2. Again, there is a variant of this argument in the case where (26) holds. Then we have (27) available, and for the same reason in (32) the upper bound can be replaced by A . With these estimates in hand, the above argument produces the better bound

$$K_N(\mathcal{P}) \ll_A \log N.$$

Once again, this confirms a claim from the introduction.

The verification of the lower bound in Theorem 2 is somewhat more complex. Throughout the argument below we use the notation as introduced in Lemma 4. Let $N \geq N_0(q)$, and consider a number $n \in \mathcal{B}(N)$. Then by (8) and the triangle inequality,

$$B\varphi(q) \log N \leq |\Delta_n| \leq 33C\varphi(q) \log N.$$

Furthermore, one has $p_{n+2} - p_{n+1} \leq p_{n+2} - p_n \leq 33C\varphi(q) \log N$, and the same inequality holds for $p_{n+1} - p_n$. Hence, by Lemma 3 and (30), we see that $\Gamma_n \leq (44C)^2 N^{-2}$, and that

$$\frac{B}{2CN} \frac{|\Delta_n|}{p_{n+1}} \leq \frac{44C}{N}.$$

Here we have used that N is large. These last inequalities combine with the bound on Γ_n to

$$\frac{B}{3CN} \leq \left| \frac{\Delta_n}{p_{n+1}} - \Gamma_n \right| \leq \frac{45C}{N} \leq \frac{1}{9}.$$

However, when $|t| \leq \frac{1}{2}$ one has $\frac{1}{2}|t| \leq |\log(1+t)| \leq 2|t|$, so that (31) now yields

$$\frac{B}{6CN} \leq \left| \log \frac{p_n p_{n+2}}{p_{n+1}^2} \right| \leq \frac{90C}{N},$$

We apply Lemma 3 again to confirm that for $j = 1$ and 2 , one has

$$1 \leq \frac{p_{n+j+1}}{p_{n+j}} = 1 + \frac{p_{n+j+1} - p_{n+j}}{p_{n+j}} \leq 1 + \frac{44C}{N} \leq \frac{10}{9}.$$

Thus

$$0 \leq \log \frac{p_{n+j+1}}{p_{n+j}} \leq \frac{1}{9},$$

and hence,

$$\frac{B}{7CN} \leq \frac{\left| \log \frac{p_{n+2}p_n}{p_{n+1}^2} \right|}{1 + \left(\log \frac{p_{n+2}}{p_{n+1}} \right) \left(\log \frac{p_{n+1}}{p_n} \right)} \leq \frac{90C}{N} \leq \frac{2}{9}.$$

For $0 \leq t \leq \frac{1}{4}$ one has $\arctan t \geq \frac{1}{2}t$. Therefore, by (29), we conclude that $k_n \geq B/(14CN)$ holds for all $n \in \mathcal{B}(N)$. By (28) and Lemma 4 it follows that

$$\sum_{n \in \mathcal{B}(N)} k_n \geq \frac{B}{28C} \geq \frac{1}{10^8 \delta_{2N}^3}.$$

In this estimate, we replace N by $2^{-j}N$ and sum over $1 \leq j \leq \frac{1}{3} \log N$. But then $2^{-j}N \geq \sqrt{N}$ for all j , and we only have to arrange that $\sqrt{N} \geq N_2(q)$, with N_2 as in Lemma 4. We conclude that $K_N(\mathcal{P}) \geq 10^{-8} \delta_N^{-3} \log N$, as required to complete the proof of Theorem 2.

5. A SCATTERED SEQUENCE

We end with a brief description of a sequence with large curvature. Let $A_1 = 10$ and define A_l by the recursion $A_{l+1} = 2A_l \log 4A_l$ ($l \geq 1$). Note that the intervals $I_l = [A_l, 4A_l]$ are disjoint. We now construct a set of primes \mathcal{Q} as follows. If the number of primes in I_l is even, then all these primes become elements of \mathcal{Q} , and in the contrary case, we put all but the smallest of the primes in I_l in \mathcal{Q} . Primes that are not in some I_l are not in \mathcal{Q} . Note that we have arranged that the number of elements in $\mathcal{Q} \cap I_l$ is even.

We claim that \mathcal{Q} is δ -dense with $\delta(x) = 1/\log x$. To see this, let $\pi_{\mathcal{Q}}(x)$ denote the number of primes in \mathcal{Q} not exceeding x , and suppose that x is large. Then, there is some l with $\frac{5}{4}A_l \leq x \leq \frac{5}{4}A_{l+1}$. In the case where $\frac{5}{4}A_l \leq x \leq 4A_l$, Chebyshev's estimates give $\pi_{\mathcal{Q}}(x) \gg x/\log x$ which is more than is required. In the range $4A_l \leq x \leq \frac{5}{4}A_{l+1}$ we use the prime number theorem to see that

$$\begin{aligned} \pi_{\mathcal{Q}}(x) &\geq \pi_{\mathcal{Q}}(4A_l) \geq \frac{3A_l}{\log 4A_l} (1 + o(1)) \\ &= \left(\frac{3}{2} + o(1) \right) \frac{A_{l+1}}{(\log A_{l+1})^2} = \left(\frac{6}{5} + o(1) \right) \frac{\frac{5}{4}A_{l+1}}{(\log \frac{5}{4}A_{l+1})^2}. \end{aligned}$$

In particular, this confirms (5) with $\delta(x) = 1/\log x$, as desired.

Let (q_j) denote the sequence of the elements of \mathcal{Q} in ascending order. Now let l be large. By construction, $\pi_{\mathcal{Q}}(4A_l)$ is even, say $2N$. Then $q_{2N} < 4A_l$ but $q_{2N+1} > A_{l+1}$. Also, by the prime number theorem, $q_{2N+2} - q_{2N+1} = o(A_{l+1})$ so that we now have $|\Delta_{2N}| \geq A_{l+1}(1 + o(1))$, and hence, again by the prime number theorem,

$$\frac{|\Delta_{2N}|}{q_{2N}} \geq \frac{A_{l+1}}{4A_l} (1 + o(1)) = \left(\frac{1}{2} + o(1) \right) \log A_l.$$

Further, the equation $2N = \pi_{\mathcal{Q}}(4A_l)$ and the straightforward bounds

$$\frac{3A_l}{\log A_l} (1 + o(1)) \leq \pi_{\mathcal{Q}}(4A_l) \leq \frac{4A_l}{\log A_l} (1 + o(1))$$

imply that $\log A_l = (1 + o(1)) \log N$, so that we arrive at

$$\frac{|\Delta_{2N}|}{q_{2N}} \geq \frac{1}{3} \log N.$$

In particular, we see that the sum considered in Theorem 3 contains a single term exceeding $\frac{1}{3} \log N$, which is of the order of δ_{2N+2}^{-1} .

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